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THE THERMOELASTIC EQUILIBRIUM OF CONICAL BODIES†

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An effective solution of various boundary-value problems of thermoelasticity for a hollow infinite cone and an infinite conical panel, when conditions of symmetry or antisymmetry are specified on the plane boundaries of the panel, is constructed in a spherical system of coordinates by the method of separation of variables. A solution of the boundary-contact problems of thermoelasticity is constructed in the case when such bodies are multilayer bodies. The contact surfaces are conical surfaces. A steady temperature field and surface perturbations act on the body. Moreover, certain boundary-value problems of the theory of elasticity are solved by this method for bodies bounded by coordinate surfaces of a spherical system of coordinates, when inhomogeneous boundary conditions are specified on the conical surfaces of the body, and conditions of symmetry or antisymmetry are specified on the plane boundaries, while special homogeneous boundary conditions are specified on the spherical surfaces.

The elastic equilibrium of infinite conical bodies has been investigated by many researchers (see the brief review of these publications in [1]). The problem of the elastic equilibrium of a hollow infinite cone was solved in [1]. Apparently, even the problem of the elastic equilibrium of a hollow infinite cone had not previously been solved.

1. FORMULATION OF THE PROBLEM

We consider and solve two classes of problems in a spherical system of coordinates r, α, β:

(1) static problems of thermoelasticity for an infinite conical panel

$$\Omega = \{0 < r < \infty, 0 < \alpha < \alpha_1, \beta_0 < \beta < \beta_1\} \tag{1.1}$$

and for a multilayered infinite conical panel

$$\Omega^* = \Omega_1 + \Omega_2 + \Omega_3 + \dots$$

where

$$\Omega_k = \{0 < r < \infty, 0 < \alpha < \alpha_1, \beta_k < \beta < \beta_{k+1}\}, \quad k = 0, 1, 2, \dots \tag{1.2}$$

and conditions of symmetry or antisymmetry [2] are specified here and henceforth when $\alpha = \alpha_j$ ($j = 0, 1; \alpha_0 = 0$);

(2) certain boundary-value problems of the theory of elasticity for a finite conical panel

$$\Omega = \{r_0 < r < r_1, 0 < \alpha < \alpha_1, \beta_0 < \beta < \beta_1\} \tag{1.3}$$

when special conditions are specified for $r = r_j$ ($j = 0, 1$).

In both classes of problems a perturbation is introduced from the conical surfaces $\beta = \beta_0$ and $\beta = \beta_1$, while in the case of a multilayered body it is also introduced from the conical contact surfaces. For an infinite conical panel the boundary conditions and temperature perturbations on the conical surfaces $\beta = \beta_0$ and $\beta = \beta_1$ can be arbitrary, like the contact conditions in the case of the thermoelastic equilibrium of a multilayered panel.

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2. THE EQUILIBRIUM EQUATIONS, THE BOUNDARY CONDITIONS AND THE GENERAL SOLUTION

The equations of thermoelastic equilibrium for a uniform isotopic body can be represented in the form [3]

$$\Delta \mathbf{U} - \frac{2}{\kappa-2}\operatorname{grad}\operatorname{div}\mathbf{U} - \frac{8-\kappa}{\kappa-2}\gamma \operatorname{grad} T = 0, \quad \Delta T = 0; \quad \kappa = 4(1-\nu) \tag{2.1}$$

Here $\mathbf{U} = u_1\mathbf{l}_1 + u_2\mathbf{l}_2 + u_3\mathbf{l}_3$ is the vector of the displacements along the axes of a Cartesian system of coordinates, T is the change in the temperature of the body, ν is Poisson's ratio and γ is the coefficient of linear thermal expansion.

It follows from the first equation of (2.1) that $\Delta\operatorname{div}\mathbf{U} = 0$, and hence

$$2\operatorname{grad}T = \Delta(\mathbf{R}, T), \quad 2\operatorname{grad}\operatorname{div}\mathbf{U} = \Delta(\mathbf{R}\operatorname{div}\mathbf{U}), \quad \mathbf{R} = x\mathbf{l}_1 + y\mathbf{l}_2 + z\mathbf{l}_3$$

Using these equations, the first equation of (2.1) can be written in the form

$$\Delta[2(\kappa-2)\mathbf{U} + \mathbf{R}(2\operatorname{div}\mathbf{U} - (8-\kappa)\gamma T)] = 0 \tag{2.2}$$

and also in the form

$$\operatorname{grad}[\kappa\operatorname{div}\mathbf{U} - (8-\kappa)\gamma T] - (\kappa-2)\operatorname{rot}\operatorname{rot}\mathbf{U} = 0 \tag{2.3}$$

(we have used the well-known identity $\Delta\mathbf{U} = \operatorname{grad}\operatorname{rot}\mathbf{U} - \operatorname{rot}\operatorname{rot}\mathbf{U}$). In a spherical system of coordinates (or in any other form of rotationally symmetrical coordinates) we thus obtain the equation

$$\Delta[(\kappa-2)H\operatorname{rot}^{(\alpha)}\mathbf{U} + z(\kappa\operatorname{div}\mathbf{U} - (8-\kappa)\gamma T)] = 0 \tag{2.4}$$

where $\operatorname{rot}^{(\alpha)}\mathbf{U}$ is the projection of the vector $\operatorname{rot}\mathbf{U}$ onto the tangent to the coordinate line α, while H is one of the three Lamé coefficients in rotationally symmetrical coordinates (the remaining two Lamé coefficients are equal to one another), which is equal to $r\sin\beta$ in a spherical system of coordinates.

Finally, using relations (2.2)–(2.4), the following equations can be obtained in a spherical system of coordinates r, α, β

$$\begin{aligned}
&\Delta K = 0, \quad r^2\sin^2\beta\Delta D - D - 2v_\alpha = 0, \quad r^2\sin^2\beta\Delta v - v + 2D_\alpha = 0\\
&w_{\alpha\alpha} + \kappa^2 w = \sin\beta(v_\alpha - \kappa D)_\beta - (\kappa-1)\cos\beta(v_\alpha - \kappa D) -\\
&-\kappa r\sin\beta(K_r + \kappa r^{-1}K) - \tfrac{1}{2}(8-\kappa)\gamma r\sin\beta[\sin\beta T_\beta + \cos\beta(rT)_r]\\
&u_{\alpha\alpha} + \kappa^2 u = r\sin\beta[(v_\alpha - \kappa D)_r - (\kappa-1)r^{-1}(v_\alpha - \kappa D)] +\\
&+\kappa\sin\beta K_\beta + \kappa^2 K\cos\beta - \tfrac{1}{2}(8-\kappa)\gamma r[r\sin^2\beta T_r - \sin\beta(\cos\beta T)_\beta - \kappa T]
\end{aligned} \tag{2.5}$$

where

$$K = u\cos\beta - w\sin\beta - \frac{\sin\beta}{\kappa}[u_\beta - (rw)_r] - \frac{8-\kappa}{2\kappa}\gamma r\cos\beta T$$

$$D = u\sin\beta + w\cos\beta + \frac{1}{\kappa-2}\left[\frac{\sin\beta}{r}(r^2u)_r + (w\sin\beta)_\beta + v_\alpha\right] - \frac{8-\kappa}{2(\kappa-2)}\gamma r\sin\beta T$$

Here, u, v, w are the components of the displacement vector $\mathbf{U}$ along the tangents to the coordinate lines r, α, β, and the subscripts, r, α and β denote partial derivatives with respect to the corresponding coordinates.

The expressions for the stresses in terms of the displacements and the temperature have the form

$$\frac{1}{\mu}A^{(\alpha)} = \chi_2 \mathrm{div}\mathbf{U} + \frac{2}{r\sin\beta}(\upsilon_\alpha + w\cos\beta + u\sin\beta) - \chi_3\gamma T$$

$$\frac{1}{\mu}R^{(r)} = \chi_2 \mathrm{div}\mathbf{U} + 2u_r - \chi_3\gamma T, \quad \frac{1}{\mu}B^{(\mu)} = \chi_2 \mathrm{div}\mathbf{U} + \frac{2}{r}(w_\beta + u) - \chi_3\gamma T =$$

$$= (\kappa-2)\left(\frac{\chi_2}{r\sin\beta}D + \frac{2}{\kappa-2}\frac{1}{r}w_\beta + \frac{1}{r}u - \frac{\chi_2}{r}\mathrm{ctg}\beta w - \frac{1}{2}\chi_3\gamma T\right) \tag{2.6}$$

$$\frac{1}{\mu}B^{(\alpha)} = \frac{1}{r\sin\beta}\left[w_\alpha + \sin^2\beta\left(\frac{\upsilon}{\sin\beta}\right)_\beta\right], \quad \frac{1}{\mu}B^{(r)} = \frac{1}{r}(u_\beta + rw_r - w)$$

$$\frac{1}{\mu}R^{(\alpha)} = r\left(\frac{\upsilon}{r}\right)_r + \frac{1}{r\sin\beta}\upsilon_\alpha, \quad \chi_n = \frac{2^n - \kappa}{\kappa - 2}, \quad \mu = \frac{E}{2(1+\nu)}$$

where $R^{(r)}, A^{(\alpha)}, B^{(\beta)}$ are the normal stresses, $A^{(\beta)} = B^{(\alpha)}, R^{(\beta)}, A^{(r)} = R^{(\alpha)}$ are the shear stresses and E is the modulus of elasticity.

We will henceforth consider the thermoelastic equilibrium of an infinite conical panel, occupying region (1.1).

The boundary conditions, which will occur here, are given below

$$\begin{aligned} \alpha = \alpha_j\!: \ &\text{a) } T_\alpha = 0, \quad \upsilon = 0, \quad B^{(\alpha)} = 0, \quad R^{(\alpha)} = 0 \quad \text{or} \\ &\text{b) } T = 0, \quad A^{(\alpha)} = 0, \quad w = 0, \quad u = 0 \end{aligned} \tag{2.7}$$

$$\beta = \beta_j\!: \theta_{j1}T_\beta + \theta_{j2}T = \tau_j(r,\alpha) \tag{2.8}$$

$$\begin{aligned} &\beta = \beta_j\!: g_{j1}B^{(\beta)} + g_{j2}\frac{w}{r} = f_{j1}(r,\alpha), \quad g_{j3}B^{(r)} + g_{j4}\frac{u}{r} = f_{j2}(r,\alpha) \\ &g_{j5}B^{(\alpha)} + g_{j6}\frac{\upsilon}{r} = f_{j3}(r,\alpha) \end{aligned} \tag{2.9}$$

Here $j = 0, 1$, and $\alpha_0 = 0$; $\theta_{j1}, \theta_{j2}, g_{jl}, \ldots, g_{j6}$ are specified constants, subject to the conditions

$$\theta_{j1}\theta_{j2} \ge 0, \quad g_{j1}g_{j2} \ge 0, \quad g_{j3}g_{j4} \ge 0, \quad g_{j5}g_{j6} \ge 0$$

We will consider the conditions imposed on the function $\tau_i(\mathrm{r},\alpha)$ and $f_{jk}(r,\alpha)$ $(k = 1, 2, 3)$ in Section 3 (when solving specific boundary-value problems); we will merely indicate here that these functions are such that the matching conditions are satisfied on the edges of the infinite conical panel. Conditions (2.9) can be assumed to be non-classical if at least seven of the 12 coefficient g_{jk} $(k = 1, 2, \ldots, 6)$ are non-zero. When $g_{j2} = g_{j4} = g_{j6} = 0$ and $g_{j1} = g_{j3} = g_{j5} = 1$ we will have the boundary conditions of the first problem of thermoelasticity, and when $g_{j1} = g_{j3} = g_{j5} = 0$ and $g_{j2} = g_{j4} = g_{j6} = 1$ we will have the second problem of thermoelasticity, etc. Note that it follows from the form of conditions (2.8) and (2.9) that some conditions can be specified when $\beta = \beta_0$, and others when $\beta = \beta_1$.

Conditions (2.7) in case *a* are called symmetry conditions, while in case *b* they are called antisymmetry conditions.

It can be shown by a direct check that the solution of the system consisting of the first three equations of (2.5) can be represented in the form

$$K = \varphi_1, \quad D = (\varphi_2)_\beta - \mathrm{ctg}\beta(\varphi_3)_\alpha, \quad \upsilon = (\varphi_3)_\beta + \mathrm{ctg}\beta(\varphi_2)_\alpha \tag{2.10}$$

where φ_1, φ_2 and φ_3 are arbitrary harmonic functions.

Using the method of separation of variables and taking boundary conditions (2.7)–(2.9) into account, the harmonic functions $\mathrm{T} \equiv \varphi_0$, φ_1, φ_2 and φ_3 can be represented in the form

$$\varphi_j = r^{-1/2}\sum_{\tilde{m}=0}^{\infty}\int_0^{\infty} L(\Phi^{(j)})dn, \quad j = 0, 1, 2, 3$$

$$L(\Phi^{(j)}) = \Phi_1^{(j)}(\beta, m, n)\cos(m\alpha)\cos(n\ln r) + \Phi_2^{(j)}(\beta, m, n)\cos(m\alpha)\sin(n\ln r) + \\ + \Phi_3^{(j)}(\beta, m, n)\sin(m\alpha)\cos(n\ln r) + \Phi_4^{(j)}(\beta, m, n)\sin(m\alpha)\cos(n\ln r) \tag{2.11}$$

The functions $\Phi_k^{(j)}$ (β, m, n) can be expressed in terms of Legendre functions as follows:

$$\Phi_k^{(j)}(\beta, m, n) = A_{kmn}^{(j)}\frac{P_{-1/2+in}^{-m}(\cos\beta)}{P_{-1/2+in}^{-m}(\cos\beta_1)} + B_{kmn}^{(j)}\frac{P_{-1/2+in}^{-m}(-\cos\beta)}{P_{-1/2+in}^{-m}(-\cos\beta_0)}, \\ k = 1, 2, 3, 4; \quad i = \sqrt{-1} \tag{2.12}$$

Here $A_{kmn}^{(j)}$, $B_{kmn}^{(j)}$ are constants. If $0 \leqslant \alpha \leqslant 2\pi$, then instead of boundary conditions (2.7) we will have the periodicity condition, where $m = \tilde{m}(\tilde{m} = 0, 1, 2, \ldots)$ and, generally speaking, all four functions $\Phi_k^{(j)}$ (β, m, n) will be non-zero; if $0 \leqslant \alpha \leqslant \alpha_1 \leqslant 2\pi$, then $m = \pi\tilde{m}/\alpha$ or $m = \pi(\tilde{m} + 1/2)\alpha_1$ and either $\Phi_3^{(j)}$ (β, m, n) and $\Phi_4^{(j)}$ (β, m, n) or $\Phi_1^{(j)}$ $(\beta, m, n) = 0$ and $\Phi_2^{(j)}$ $(\beta, m, n) = 0$. If m is a half-integer number, i.e. a number of the form $\pm(\tilde{m} + 1/2)$, then instead of the functions which occur in expression (2.12) we can use the elementary functions $P_{-1/2+in}^{-\tilde{m}-1/2}$ $(\cos\beta)$ and $P_{-1/2+in}^{\tilde{m}+1/2}$ $(\cos\beta)$, or more correctly, their combinations. The expressions for the functions occurring in relation (2.12) with arbitrary index m are known from the reference literature [4, p. 148]. When $\beta_0 = 0$, we will take $B_{kmn}^{(j)} = 0$ for the boundedness of the solution.

Substituting expressions (2.11) into relations (2.10), we will have

$$q = r^{-1/2}\sum_{\tilde{m}=0}^{\infty}\int_0^{\infty} L(F^{(q)})dn, \quad q = K, D, \upsilon \tag{2.13}$$

where $F_k^{(q)}$ (β, m, n) are known expressions, which include the constant $A_{kmn}^{(j)}$ and $B_{kmn}^{(j)}$ from (2.12) and which depend on β, m and n (the operator $L(F^{(q)})$ is obtained from the operator $L(\Phi_k^{(j)})$ by replacing $\Phi_k^{(j)}$ (β, m, n) by $F_k^{(q)}$ (β, m, n) in the latter. Taking expressions (2.13) into account, the fourth and fifth equations of (2.5) take the form

$$p_{\alpha\alpha} + \kappa^2 p = r^{-1/2}\sum_{\tilde{m}=0}^{\infty}\int_0^{\infty}[L(F^{(q)}) - \Gamma_p(\varphi_{0mn})]dn, \quad p = w, u \tag{2.14}$$

Here

$$\Gamma_u(\varphi_{0mn}) = \frac{1}{2}(8-\kappa)\gamma r[r\sin^2\beta(\varphi_{0mn})_r - \sin\beta(\cos\beta\varphi_{0mn})_\beta - \kappa\varphi_{0mn}]$$

$$\Gamma_w(\varphi_{0mn}) = \frac{1}{2}(8-\kappa)\gamma r\sin\beta[\sin\beta(\varphi_{0mn})_\beta + \cos\beta(r\varphi_{0mn})_r]$$

where φ_{0nm} is the expression under the series and integral signs in (2.11) when $j = 0$, and $L(F^{(p)})$ is an expression similar to the expression for $L(F^{(q)})$ from relation (2.13).

We will assume that $\kappa \neq m$, and hence the solutions of Eqs (2.14) can be written in the form

$$p = r^{-1/2}\sum_{\tilde{m}=0}^{\infty}\int_0^{\infty}\frac{1}{\kappa^2 - m^2}[L(F^{(p)}) - \Gamma_p(\varphi_{0mn})]dn, \quad p = w, u$$

Consequently, we finally have for K, D, v, w and u

$$q = r^{-1/2}\sum_{\tilde{m}=0}^{\infty}\int_0^{\infty}[L(F^{(q)}) - P_1^{(q)}(\varphi_{0mn})]\kappa_m dn, \quad q = K, D, \upsilon, w, u \tag{2.15}$$

where $P_1^{(q)} = 0$ when $q = K, D, \upsilon$, $P_1^{(q)}(\varphi_{0mn}) = \Gamma_p(\varphi_{0mn})$ when $p = w, u$, $\kappa_m = 0$ when $q = K, D, \upsilon$ and $\kappa_m = 1/(\kappa^2 - m^2)$ when $q = w, u$. Using relations (2.15) and (2.6), we obtain for the stresses

$$\sigma = r^{-3/2} \sum_{\tilde{m}=0}^{\infty} \int_0^{\infty} [L(F^{(\sigma)}) - P_2^{(\sigma)}(\varphi_{0mn})] dn, \quad \sigma = R^{(r)}, A^{(\alpha)}, B^{(\beta)}, B^{(\alpha)}, B^{(r)}, R^{(\alpha)} \tag{2.16}$$

where $P_2^{(\sigma)}(\varphi_{0mn})$ is a known expression, which depends on φ_{0mn}.

3. AN ANALYTICAL SOLUTION OF SOME BOUNDARY-VALUE PROBLEMS

As an example, we will construct a regular solution of boundary-value problem (2.5), (2.7b) and (2.9) in region (1.1) for $\beta = \pi - \beta_0$ and for $T = 0$ with

$$g_{j2} = g_{j4} = g_{j6} = 0, \quad g_{j1} = g_{j3} = g_{j5} = 1$$

We will call this problem problem G_0 and we will represent it in the form

$$G_0 = G_1 + G_2$$

where G_1 is problem G_0 with a load distributed symmetrically about the plane $\beta = \pi/2$, while G_2 is problem G_0 with a load distributed antisymmetrically about the plane $\beta = \pi/2$.

We spoke above of a regular solution of the problem, and hence we will define the meaning of regularity.

A solution of system (2.5), defined by the functions u, v, w, will be called regular if the functions u, v, w are thrice continuously differentiable in the region $\tilde{\Omega}$, where $\tilde{\Omega}$ is the region Ω together with the boundaries $\alpha = \alpha_j$, while on the surface $\beta = \beta_j$ they can be represented, together with their first derivatives, by a certain absolutely and uniformly converging expression, representing a Fourier–Mellin integral with respect to r and a Fourier series with respect to α.

We will confine ourselves to solving boundary-value problem G_1, in which in addition to conditions (2.7, b), the symmetry conditions

$$w = 0, \quad B^{(r)} = 0, \quad B^{(\alpha)} = 0 \quad \text{when} \quad \beta = \pi/2$$

and the conditions

$$B^{(\beta)} = f_1(r,\alpha), \quad B^{(r)} = f_2(r,\alpha), \quad B^{(\alpha)} = f_3(r,\alpha) \quad \text{when} \quad \beta = \beta_0 \tag{3.1}$$

are satisfied. The functions φ_1, φ_2 and φ_3 in the case considered take the form

$$\varphi_k = r^{-1/2} \sum_{\tilde{m}=1}^{\infty} \int_0^{\infty} R^{(ks)} P^{(1)} dn, \quad \varphi_3 = r^{-1/2} \sum_{\tilde{m}=0}^{\infty} \int_0^{\infty} R^{(3c)} P^{(2)} dn$$

Here

$$P^{(k)} = \frac{P^{-m}_{-1/2+in}(-\cos\beta) + (-1)^k P^{-m}_{-1/2+in}(\cos\beta)}{P^{-m}_{-1/2+in}(-\cos\beta_0)}$$

$$R^{(ks)} = R^{(k)}\sin(m\alpha), \quad R^{(3c)} = R^{(3)}\cos(m\alpha), \quad R^{(j)} = A^{(j)}_{1mn}\cos(n\ln r) + A^{(j)}_{2mn}\sin(n\ln r)$$

$k = 1, 2$, $j = 1, 2, 3$, $m = \pi\tilde{m}/\alpha_1$

The constants $A^{(1)}_{1mn}, A^{(1)}_{2mn}, \ldots, A^{(3)}_{2mn}$ are found from the system of equations

$$\begin{aligned} F_3^{(\beta\beta)}(\beta_0, m, n) &= f^{(c)}_{1mn}, \quad F_4^{(\beta\beta)}(\beta_0, m, n) = f^{(s)}_{1mn} \\ F_3^{(\beta r)}(\beta_0, m, n) &= f^{(c)}_{2mn}, \quad F_4^{(\beta r)}(\beta_0, m, n) = f^{(s)}_{2mn} \\ F_1^{(\beta\alpha)}(\beta_0, m, n) &= f^{(c)}_{3mn}, \quad F_2^{(\beta\alpha)}(\beta_0, m, n) = f^{(s)}_{3mn} \end{aligned} \tag{3.2}$$

For convenience in writing formulae (3.2) we have denoted $B^{(\beta)}$, $B^{(r)}$ and $B^{(\alpha)}$ by $\beta\beta$, βr, and $\beta\alpha$ respectively, f_{kmn} and f_{kmn} are the Fourier coefficients of the functions $r^{3/2} f_k(r, \alpha)$ $(k = 1, 2, 3)$, represented by a Fourier–Mellin integral with respect to r and a Fourier series with respect to α; with respect to the functions $F_3^{(\beta\beta)}(\beta_0, m, n), \ldots, F_2^{(\beta\alpha)}(\beta_0, m, n)$, see formulae (2.16). The following conditions are imposed on the continuous functions $f_k(r, \alpha)$, which occur in conditions (3.1) [5]:

1) the derivatives of the function $f_k(r, \alpha)$ with respect to r and with respect to α are continuous and belong to the Hölder class,

2) for any value of $\alpha \in [0, \alpha_1]$ the following integral converges

$$\int_0^\infty |f_k(r, \alpha)| r^{-3/2} dr$$

It can be seen that (3.2) is a system of linear algebraic equations with a sixth-order matrix (this system can be converted appropriately for best conditionality of its matrix). The uniqueness of the solution obtained can be proved using the energy integral, and it follows from the uniqueness of the solution that the determinant system (3.2) is non-zero.

Hence, we have obtained a regular solution of problem G_1. In exactly the same way we can also solve problem G_2 and any of the boundary-value problems (2.5) and (2.7)–(2.9) for any β_0 and β_1.

In this paper we direct our main attention to obtaining regular solutions of boundary-value problems of thermoelasticity for an infinite conical panel, and will not investigate the behaviour of the solutions in the neighbourhood of $r = 0$, and when $r \to \infty$ (we will assume that functions specified on the conical boundary surfaces satisfy the requirements which guarantee that regular solutions will be obtained).

4. A MULTILAYERED INFINITE CONICAL PANEL

We will consider an infinite conical panel, multilayered along the coordinate β axis, which occupies the region Ω^*, representing the union of regions (1.2) when $k = 0, 1, 2, \ldots 3, s - 1$, which are in contact with one another along the conical surfaces $\beta = \beta_j$ $(j = 1, 2, \ldots, s - 1$, where s is the number of layers). Each layer has its own elastic and thermal characteristics. When $\alpha = \alpha_0$ and $\alpha = \alpha_1$ one of the sets of conditions (2.7) is simultaneously satisfied for all the layers.

At the boundary of the body occupying the region Ω^*, conditions (2.7)–(2.9) are satisfied with β_1 replaced by β_s in (2.8) and (2.9). The following conditions are specified on the contact surfaces $\beta = \beta_j$ $(j = 1, 2, \ldots, s - 1)$

$$\begin{aligned}
T_j - T_{j+1} &= \tau_{j1}(r, \alpha), \quad \lambda_j^*(T_j)_\beta - \lambda_{j+1}^*(T_{j+1})_\beta = \tau_{j2}(r, \alpha) \\
w_j - w_{j+1} &= \tilde{F}_{j1}(r, \alpha), \quad B_j^\beta - B_{j+1}^\beta = \tilde{F}_{j2}(r, \alpha) \\
u_j - u_{j+1} &= \tilde{F}_{j3}(r, \alpha), \quad B_j^\beta - B_{j+1}^\beta = \tilde{F}_{j4}(r, \alpha) \\
\upsilon_j - \upsilon_{j+1} &= \tilde{F}_{j5}(r, \alpha), \quad B_j^\beta - B_{j+1}^\beta = \tilde{F}_{j6}(r, \alpha)
\end{aligned}$$

Here λ_j^*, λ_{j+1}^* are the thermal conductivities and $\tau_{j1}(r, \alpha)$, $\tau_{j2}(r, \alpha)$, $\tilde{F}_{j1}(r, \alpha), \ldots, \tilde{F}_{j6}(r, \alpha)$ are given functions.

Formulating the problem of finding the thermoelastic equilibrium of a multilayered infinite conical panel, for the jth layer we will construct expressions for φ_j $(j = 0, 1, 2, 3)$, q and σ (see formula (2.11), (2.15) and (2.16)), taking conditions (2.7) into account, and, following the method described in Section 3, we obtain two systems, similar to system (3.2): one, consisting of $4s$ linear algebraic equations with $4s$ unknowns (determining the temperature field), and the other, consisting of $12s$ linear algebraic equations with $12s$ unknowns (for a body, closed with respect to α, the orders of the matrices of the corresponding systems of linear equations are equal to $8s$ and $24s$). The convergence of the series, representing the displacements and stresses, is proved and also the uniqueness of the solutions obtained.

In addition, we should also consider a number of other contact conditions, for which solutions of the boundary-contact problems of thermoelasticity for a multilayered infinite conical panel can be written just as effectively.

5. ELASTIC EQUILIBRIUM OF A FINITE CONICAL PANEL

Consider the elastic equilibrium of a finite conical panel, occupying region (1.3) when $r_0 > 0$, when conditions (2.7) remain in force (naturally, when there is no condition imposed on T), and when $r = r_j$ and $\beta = \beta_j$ one of the conditions from the following set is satisfied

$$\begin{aligned} r = r_j\text{: a) } & u = 0, \quad rv_r - (\kappa - 1)v = 0, \quad rw_r - (\kappa - 1)w = 0 \\ \text{b) } & ru_r + \kappa u = 0, \quad v = 0, \quad w = 0 \end{aligned} \tag{5.1}$$

$$\begin{aligned} \beta = \beta_j\text{: a) } & w = \tilde{f}_{j1}(r,\alpha), \quad u = \tilde{f}_{j2}(r,\alpha), \quad v = \tilde{f}_{j3}(r,\alpha) \\ \text{b) } & B^{(\beta)} = f_{j1}(r,\alpha), \quad u = \tilde{f}_{j2}(r,\alpha), \quad v = \tilde{f}_{j3}(r,\alpha) \\ \text{c) } & B^{(\beta)} = f_{j1}(r,\alpha), \quad u = \tilde{f}_{j2}(r,\alpha), \quad B^{(\alpha)} = f_{j3}(r,\alpha) \\ \text{d) } & w = \tilde{f}_{j1}(r,\alpha), \quad B^{(r)} = f_{j2}(r,\alpha), \quad B^{(\alpha)} = f_{j3}(r,\alpha) \\ \text{e) } & w = \tilde{f}_{j1}(r,\alpha), \quad B^{(r)} = \tilde{f}_{j2}(r,\alpha), \quad v = \tilde{f}_{j3}(r,\alpha) \\ \text{f) } & w = \tilde{f}_{j1}(r,\alpha), \quad u = \tilde{f}_{j2}(r,\alpha), \quad B^{(\alpha)} = f_{j3}(r,\alpha) \end{aligned} \tag{5.2}$$

We will assume that the function $r^{1/2}\tilde{f}_{jk}(r, \alpha)$, together with its first derivatives, and the functions $r^{3/2}f_{jk}(r, \alpha)$ can be expanded in uniformly converging double trigonometric series.

As an example we will consider the solution of boundary-value problem (2.5), (5.1, b), (2.7, b), (5.2, c), where in condition (5.2, c)

$$\tilde{f}_{j2}(r_j, \alpha) = 0, \quad \left[\frac{\partial}{\partial r}\tilde{f}_{j2}(r,\alpha)\right]_{r = r_j} = 0$$

The functions φ_1, φ_2 and φ_3 in the case considered take the form

$$\varphi_j = r^{-1/2}\sum_{\tilde{m}=1}^{\infty}\sum_{\tilde{n}=1}^{\infty}\Phi^{(j)}(\beta, m, n)\Psi^{(j)}_{mm}(r,\alpha), \quad j = 1, 2, 3 \tag{5.3}$$

The functions $\Phi^{(j)}(\beta, m, n)$ are defined by formula (2.12)

$$\Psi^{(1)}_{mn}(r,\alpha) = \sin(m\alpha)[(2\kappa - 1)\chi_s(r) - 2n\chi_c(r)]$$

$$\Psi^{(2)}_{mn}(r,\alpha) = \sin(m\alpha)\chi_s(r), \quad \Psi^{(3)}_{mn}(r,\alpha) = \cos(m\alpha)\chi_s(r)$$

$$m = \frac{\pi\tilde{m}}{\alpha_1}, \quad n = \frac{\pi\tilde{n}}{\ln r_1 - \ln r_0}, \quad \chi_s(r) = \sin\left(n\ln\frac{r}{r_0}\right), \quad \chi_c(r) = \cos\left(n\ln\frac{r}{r_0}\right)$$

Substituting expressions (5.3) into the last equation of (2.10) and into the last two equations of (2.5), we obtain

$$\begin{aligned} & r^{1/2}v = \sum_{\tilde{m}=0}^{\infty}\sum_{\tilde{n}=1}^{\infty}F^{(v)}(\beta, m, n)\Psi^{(3)}_{mn}(r,\alpha), \quad r^{1/2}w = \sum_{\tilde{m}=0}^{\infty}\sum_{\tilde{n}=1}^{\infty}F^{(w)}(\beta, m, n)\Psi^{(2)}_{mn}(r,\alpha) \\ & r^{1/2}u = \sum_{\tilde{m}=1}^{\infty}\sum_{\tilde{n}=1}^{\infty}F^{(u)}(\beta, m, n)\Psi^{(1)}_{mn}(r,\alpha) \end{aligned} \tag{5.4}$$

In turn, substituting expressions (5.4) into the third and fourth formulae of (2.6) and taking conditions (5.2, c) into account, we obtain,, when $\beta = \beta_j$ $(j - 0, 1)$

$$r^{-3/2}\sum_{\tilde{m}=0}^{\infty}\sum_{\tilde{n}=1}^{\infty}F^{(1)}(\beta_j, m, n)\Psi^{(2)}_{mn}(r,\alpha) = \frac{1}{\mu}f_{j1}(r,\alpha) - (\kappa - 2)r^{-1}\tilde{f}_{j2}(r,\alpha)$$

$$r^{-1/2}\sum_{\tilde m=0}^{\infty}\sum_{\tilde n=1}^{\infty}F^{(2)}(\beta_j,m,n)\Psi_{mn}^{(1)}(r,\alpha)=\tilde f_{j2}(r,\alpha) \tag{5.5}$$

$$r^{-3/2}\sum_{\tilde m=0}^{\infty}\sum_{\tilde n=1}^{\infty}F^{(3)}(\beta_j,m,n)\Psi_{mn}^{(3)}(r,\alpha)=\frac{1}{\mu}f_{f3}(r,\alpha)$$

From these relations we obtain the system

$$F^{(k)}(\beta_j,m,n)=f_{mn}^{(k)},\quad j=0,1;\quad k=1,3 \tag{5.6}$$

where $f_{mn}^{(1)}, f_{mn}^{(2)}$ and $f_{mn}^{(3)}$ are the Fourier coefficients of the functions

$$r^{3/2}\Big[\frac{1}{\mu}f_{j1}(r,\alpha)-\frac{\kappa-2}{r}\tilde f_{j2}(r,\alpha)\Big],\quad r^{1/2}\tilde f_{j2}(r,\alpha),\quad \frac{r^{3/2}}{\mu}f_{j3}(r,\alpha)$$

respectively. The constants $A_{1mn}, B_{1mn}, \ldots, B_{3mn}$ to be determined are contained in the expressions $F^{(k)}(\beta_j,m,n)$ and are determined from system (5.6) with a sixth-order matrix. We have thereby obtained a regular solution of the problem in question.

If $r_0=0$ in problem considered above, i.e. $0<r<r_1$ or $r_1<r<\infty$, the series in $\tilde n$ are replaced by an integral over n in formulae (5.3) and (5.4).

Hence, we have obtained a regular solution of the problem in region (1.3) when $t_0>0$, when $r_0=0$ and when $r_0>0$, $r_1=\infty$.

An investigation of such problems, when conditions (5.1, a) are specified on at least one of the boundary surfaces $r=r_j$, has not so far been carried out, but everything stated in Section 5 for boundary-value problems (2.5), (5.1, a), (2.7), (5.2) also.

It is not difficult to write a solution in region (1.3) in the case when one of conditions (5.1) is specified when $r=r_0$ and the other is specified when $r=r_1$.

Example 1. Consider the three-dimensional boundary-value problem of the theory of elasticity on the bending of a thin conical ring with a cut, occupying the region (1.3) when $\alpha_1=2\pi$ and $\beta_1=\pi-\beta_1$ (the faces of the cut $\alpha=0$ and $\alpha=2\pi$). The bending stressed state corresponds to the following boundary conditions

$$\beta=\beta_j\colon w=f_0r^{-1/2}\sin\Big(n_1\ln\frac{r}{r_0}\Big)\sin\frac{\alpha}{2},\quad B^{(r)}=0,\quad B^{(\alpha)}=0 \tag{5.7}$$

$$r=r_j\colon u_r+\kappa\frac{u}{r}=0,\quad \upsilon=0,\quad w=0 \tag{5.8}$$

$$\alpha=\alpha_j\colon A^{(\alpha)}=0,\quad w=0,\quad u=0 \tag{5.9}$$

where

$$j=0,1,\quad \beta_1=\pi-\beta_0,\quad \alpha_0=0,\quad \alpha_1=2\pi;\quad n_1=\frac{\pi}{\ln r_1-\ln r_0},\quad f_0=\text{const}$$

Note that in terms of the theory of thin-walled structures, conditions (5.8) and (5.9) can be interpreted as a free support.

The plane $\beta-\pi/2$ for a conical ring is a plane of asymmetry, and hence

$$B^{(\beta)}=0,\quad u=0,\quad \upsilon=0\ \text{ when }\ \beta=\pi/2 \tag{5.10}$$

and will be considered as region (1.3), in which $\beta_0<\beta<\pi/2$. In this case conditions (5.7) when $\beta=\beta_0$, like conditions (5.8) and (5.9), remain in force, and we arrive at boundary-value problem (2.5), (5.7)–(5.10) (in condition (5.7) $j=0$). In this case the functions φ_1, φ_2 and φ_3 take the form

$$\varphi_j=A_jr^{-1/2}\sin^{-1/2}\beta\,\mathrm{ch}\Big[n_1\Big(\frac{\pi}{2}-\beta\Big)\Big]\Psi_{mn}^{(0j)}(r,\alpha),\quad j=1,2,3 \tag{5.11}$$

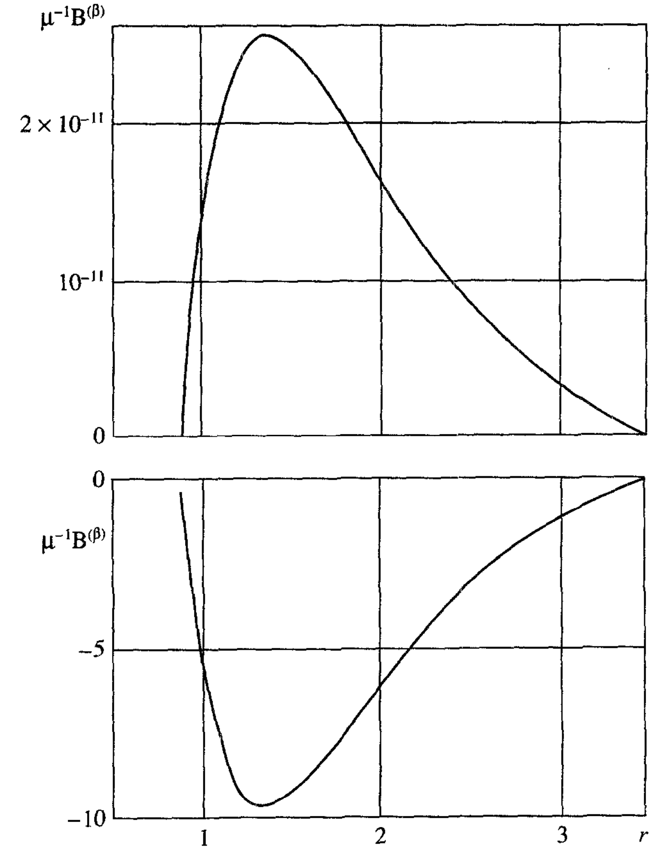

Fig. 1

where A_j are constants, and the functions $\Psi_{mn}^{(0j)}(r, \alpha)$ are equal to the corresponding functions $\Psi_{mn}^{(j)}$ (r, α) when $n = n_1$ and $m = 1/2$.

Taking

$$r_0 = \frac{15}{17}, \quad r_1 = \frac{60}{17}, \quad \kappa = 2.2\left(n_1 = \frac{\pi}{2\ln 2}, \nu = 0.45\right), \quad f_0 = 1, \quad \beta_0 = 0.45\pi$$

we find $A_1 = -0.4036$, $A_2 = 0.7765$ and $A_3 = -0.0021$, and we obtain an expression for the normal stress $B^{(\beta)}$. A graph of $B^{(\beta)}$ as a function of the coordinate r for $\alpha = \pi$ and $\beta = \pi/2$ is shown by the upper curve in Fig. 1.

Example 2. We now consider the elastic equilibrium of the conical ring with a cut of the previous example, changing only the boundary conditions on the conical surfaces, namely, we replace f_0 by $(-1)^j$ f_0 in conditions (5.7). In this case we will have compression of the conical ring, and the plane $\beta = \pi/2$ will be a plane of symmetry, so that

$$w = 0, \quad B^{(r)} = 0, \quad B^{(\alpha)} = 0 \text{ when } \beta = \pi/2 \tag{5.12}$$

and we arrive at the boundary-value problem to be considered in region (1.3) when $[0 < \beta < \pi/2]$.

If in expressions (5.11) the hyperbolic cosines are replaced by hyperbolic sines, we obtain the functions φ_1, φ_2 and φ_3 for the problem considered. Further, if r_0, r_1, κ, f_0 and β_0 remain the same as in Example 1, then $A_1 = -1.0137$, $A_2 = 1.3556$ and $A_3 = 0.0062$, and the graph of $B^{(\beta)}$ as a function of the coordinate r, when $\alpha = \pi$ and $\beta = \pi/2$, will be the lower curve in the figure.

As might have been expected, in a bent thin conical ring there is essentially no stress $B^{(\beta)}(r, \pi, \pi/2)$ (its least value is equal to 2.6×10^{-11} μ and even if we assume $\mu = 2 \times 10^6$ kg/cm^2, we obtain 5.2×10^{-5} kg/cm^3), whereas in the case of the compression of the above-mentioned ring the greatest modulus value is $B^{(\beta)}(r, \pi, \pi/2) = 9.6\mu$.

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